MTH 166

Lecture-24

Level Surfaces and Parametric Equation of a Straight Line

Unit 5: **Vector Calculus-I**

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-15**)

Topic:

Level Surfaces, Parametric equation of a Straight Line

Learning Outcomes:

1. To find Level surfaces of a given scalar function

2. To find parametric equation of a straight line

Level Surfaces:

DEFINITION The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

Let $f(x, y, z)$ be a given scalar surface.

Level surfaces corresponding to this $f(x, y, z)$ are given by: $f(x, y, z) = c$ (1)

Infact, this equation (1) gives family of surfaces that never intersect with each other

For different values of constant c, we get different members of this family of level surfaces.

Find the Level surfaces of the scalar fields defined by following functions:

Problem 1. $\boldsymbol{f = x + y + z}$

Solution. The given scalar function is:

$$f = x + y + z \qquad (1)$$

Level surfaces are given by: $f = c$

$\Rightarrow x + y + z = c$

Which is a family of Parallel planes.

Find the Level surfaces of the scalar fields defined by following functions:

Problem 2. $\boldsymbol{f = x^2 + y^2 + z^2}$

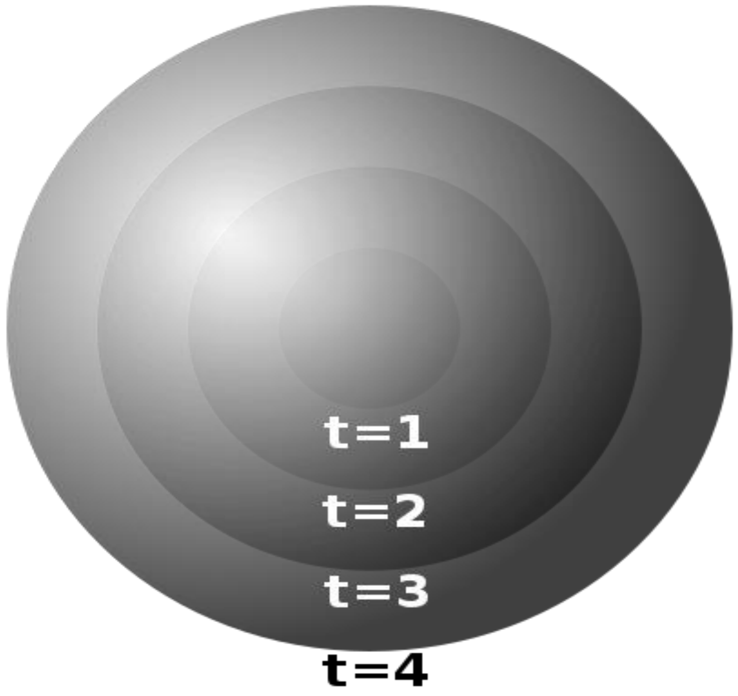
Solution. The given scalar function is:

$$f = x^2 + y^2 + z^2 \qquad (1)$$

Level surfaces are given by: $f = t$

$\Rightarrow x^2 + y^2 + z^2 = t$

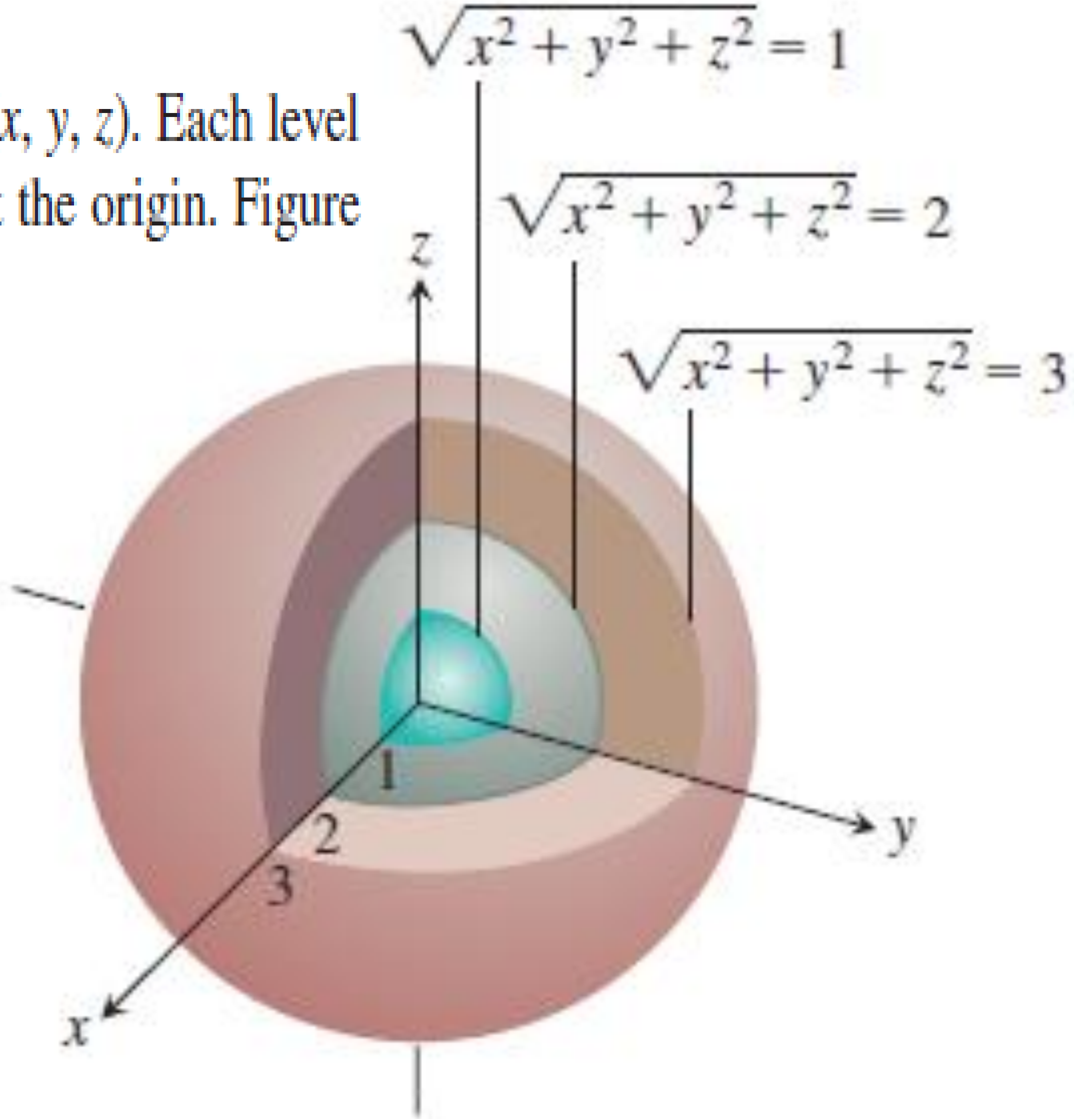
Which is a family of concentric Spheres.



Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Solution The value of f is the distance from the origin to the point (x, y, z). Each level surface $\sqrt{x^2 + y^2 + z^2} = c$, $c > 0$, is a sphere of radius c centered at the origin. Figure



Find the Level surfaces of the scalar fields defined by following functions:

Problem 3. $\boldsymbol{f = x^2 + y^2 - z}$

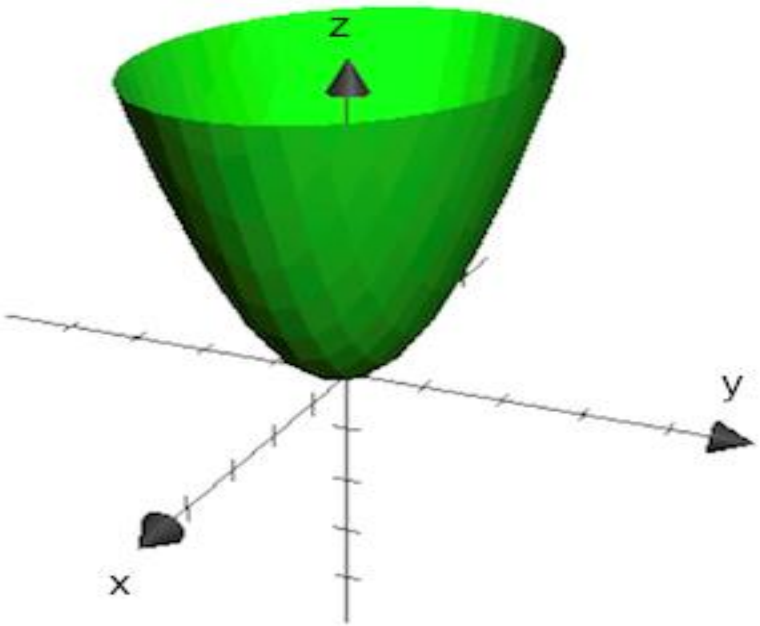
Solution. The given scalar function is:

$$f = x^2 + y^2 - z \qquad (1)$$

Level surfaces are given by: $f = c$

$\Rightarrow x^2 + y^2 - z = c$

Which is a family of Paraboloids.

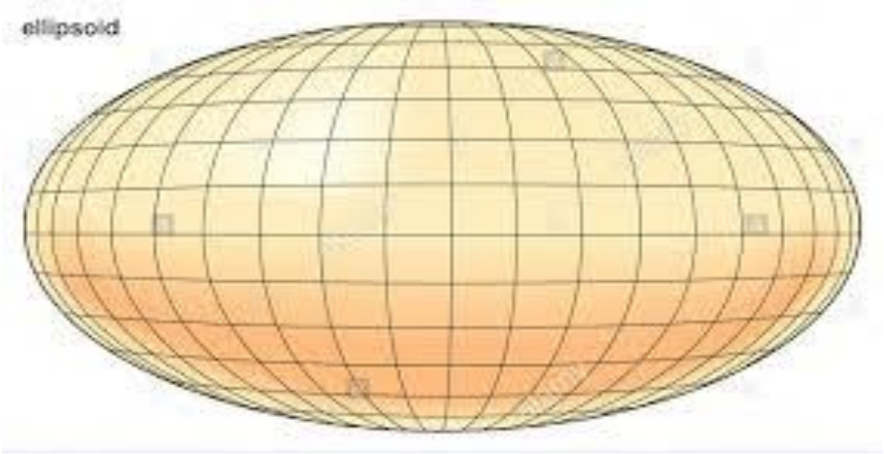


Find the Level surfaces of the scalar fields defined by following functions:

Problem 4. $\boldsymbol{f = x^2 + 9y^2 + 16z^2}$

Solution. The given scalar function is:

$$f = x^2 + 9y^2 + 16z^2 \qquad (1)$$



Level surfaces are given by: $f = c$

$\Rightarrow x^2 + 9y^2 + 16z^2 = c$

Which is a family of Ellipsoids.

Find the Level surfaces of the scalar fields defined by following functions:

Problem 5. $\boldsymbol{f = z - \sqrt{x^2 + y^2}}$

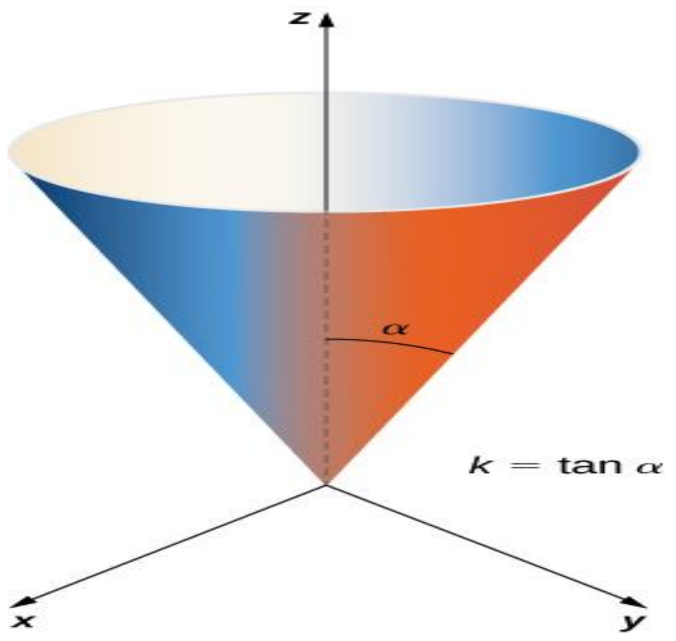
Solution. The given scalar function is:

$$f = z - \sqrt{x^2 + y^2} \quad (1)$$

Level surfaces are given by: $f = c$

$$\Rightarrow z - \sqrt{x^2 + y^2} = c$$

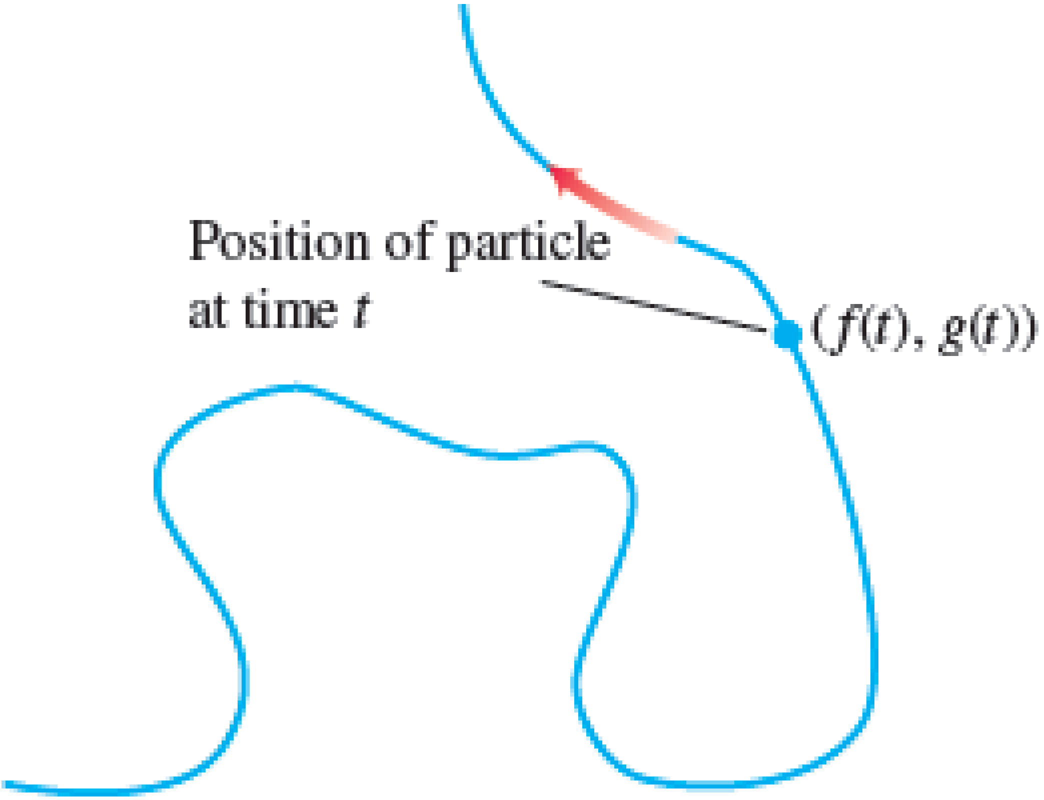
Which is a family of Cones.



DEFINITION If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

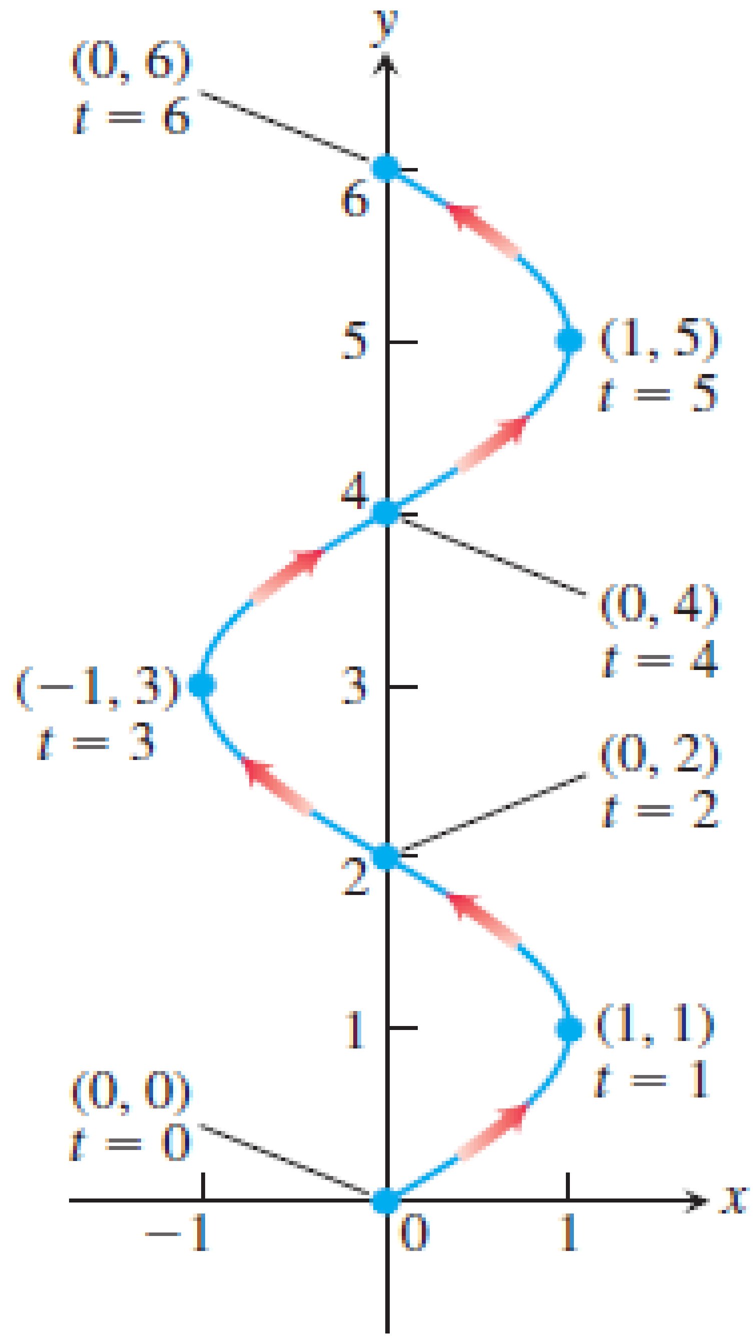


Sketch the curve defined by the parametric equations

$$x = \sin \pi t/2, \qquad y = t, \qquad 0 \le t \le 6.$$

TABLE Values of $x = \sin \pi t/2$ and $y = t$ for selected values of t.

t	x	y
0	0	0
1	1	1
2	0	2
3	−1	3
4	0	4
5	1	5
6	0	6

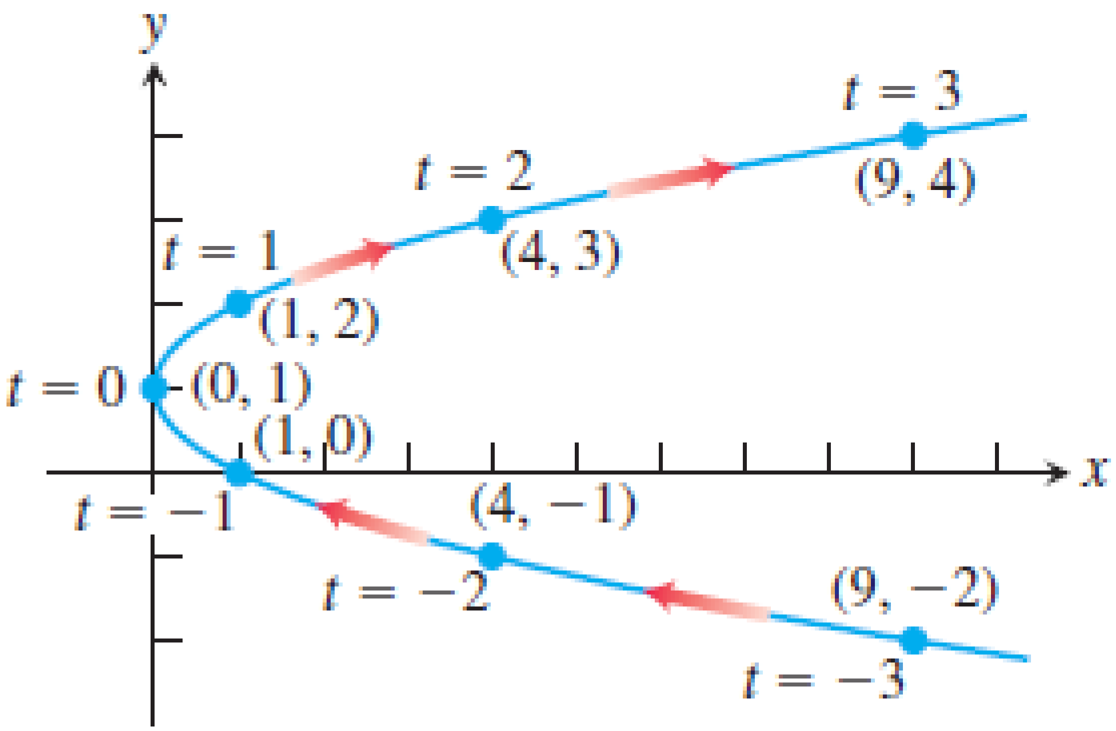


Sketch the curve defined by the parametric equations

$$x = t^2, \qquad y = t + 1, \qquad -\infty < t < \infty.$$

TABLE Values of $x = t^2$ and $y = t + 1$ for selected values of t.

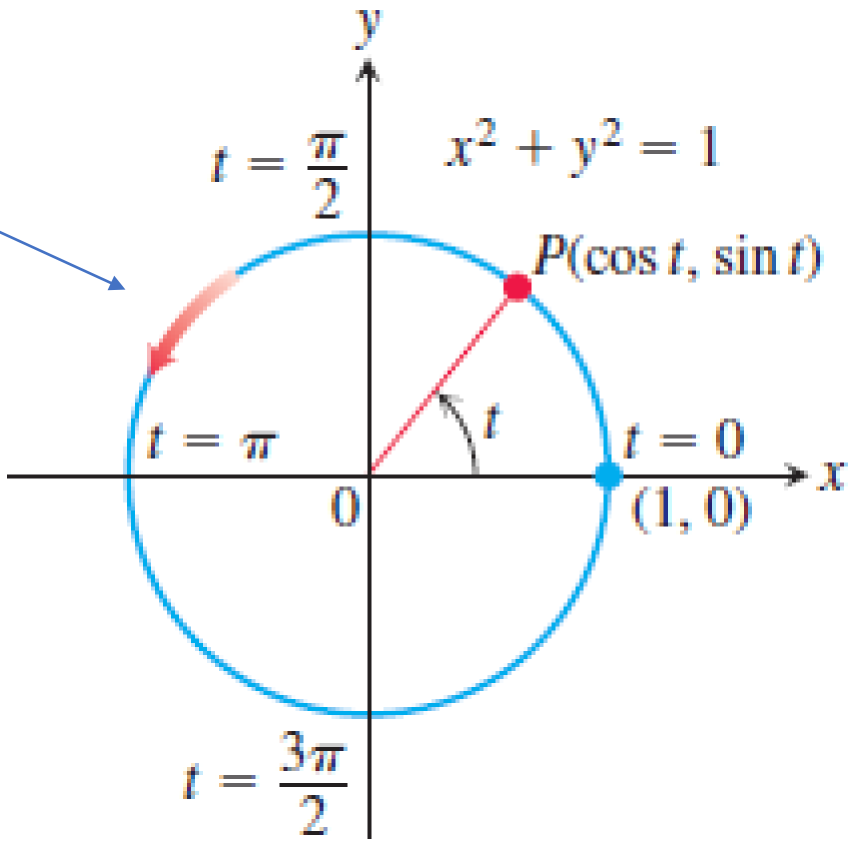
t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

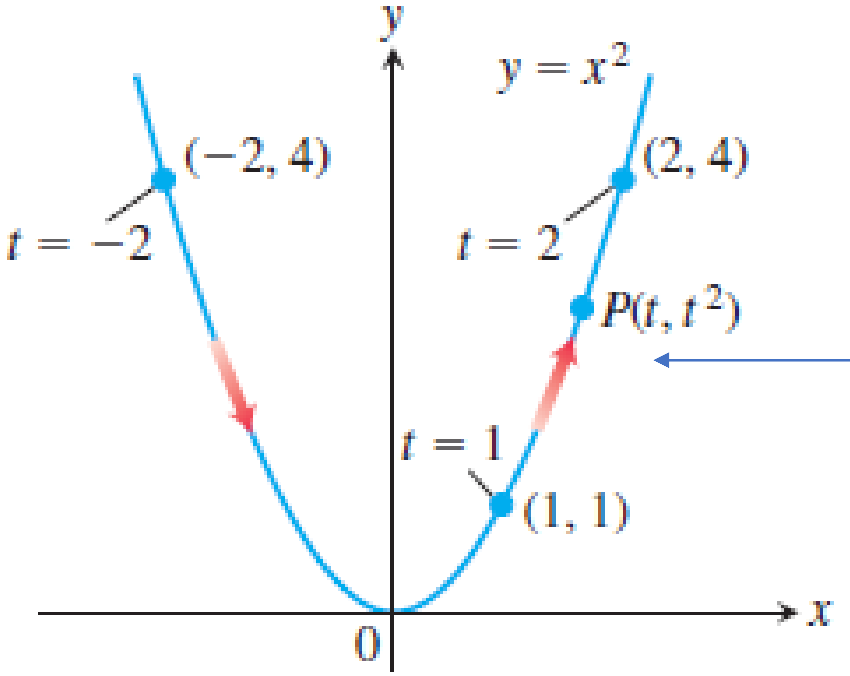


Graph the parametric curves

(a) $x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi.$

(b) $x = a\cos t, \quad y = a\sin t, \quad 0 \leq t \leq 2\pi.$





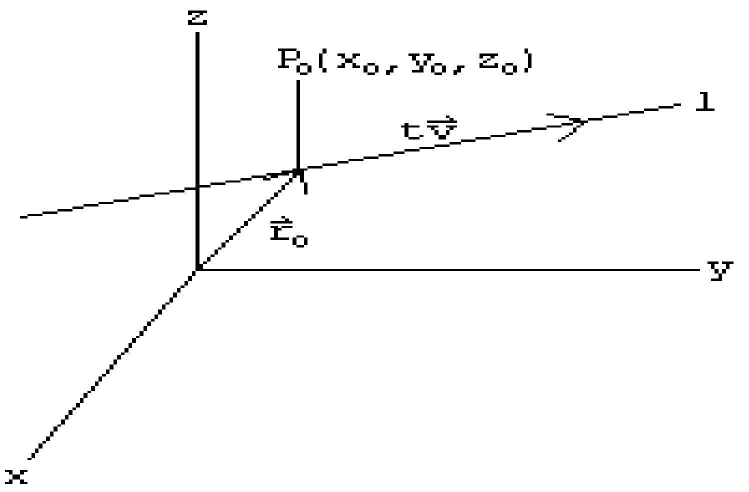
A parametrization of the graph of the function $f(x) = x^2$ is given by

$$x = t, \qquad y = f(t) = t^2, \qquad -\infty < t < \infty.$$

Parametric Equation of a Straight line passing through a point and having a given direction:

Let given point is $P_0(x_0, y_0, z_0)$ and the given direction be $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

Then Parametric Equation of a Straight line passing through point P_0 and having given direction $\vec{v}$ is given by:



$\overrightarrow{r(t)} = \overrightarrow{P_0} + t\,\vec{v}$ where t is parameter

$\Rightarrow \overrightarrow{r(t)} = (x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}) + t(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k})$

$\Rightarrow \overrightarrow{r(t)} = (x_0 + tv_1)\hat{\imath} + (y_0 + tv_2)\hat{\jmath} + (z_0 + tv_3)\hat{k}$

Find the Parametric Equation of a Straight line passing through point $\boldsymbol{P_0}$ and having direction $\vec{\boldsymbol{b}}$:

Problem 1. $P_0(1,2,3)$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Solution. Parametric Equation of a Straight line passing through point P_0 and having given direction $\vec{b}$ is given by:

$\overrightarrow{r(t)} = \overrightarrow{P_0} + t\,\vec{b}$ where t is parameter

$\Rightarrow \overrightarrow{r(t)} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

$\Rightarrow \overrightarrow{r(t)} = (1 + t)\hat{\imath} + (2 + 2t)\hat{\jmath} + (3 + 2t)\hat{k}$ **Answer.**

Find the Parametric Equation of a Straight line passing through point $\boldsymbol{P_0}$ and having direction $\vec{\boldsymbol{b}}$:

Problem 2. $P_0(1,-1,1),\ \vec{b} = \hat{\imath} - \hat{\jmath}$

Solution. Parametric Equation of a Straight line passing through point P_0 and having given direction $\vec{b}$ is given by:

$\overrightarrow{r(t)} = \overrightarrow{P_0} + t\,\vec{b}$ where t is parameter

$\Rightarrow \overrightarrow{r(t)} = (\hat{\imath} - \hat{\jmath} + \hat{k}) + t(\hat{\imath} - \hat{\jmath})$

$\Rightarrow \overrightarrow{r(t)} = (1+t)\hat{\imath} - (1+t)\hat{\jmath} + \hat{k}$ **Answer.**

Find the parametric representation of the following straight lines/curves. Use the indicated representation wherever given:

Problem 1. $x = y, y = z$

Solution. Let $z = t$ where t is a parameter

$\Rightarrow y = t \qquad (\because y = z)$

also $x = t \qquad (\because x = y)$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = t\hat{\imath} + t\hat{\jmath} + t\hat{k}$ **Answer.**

Problem 2. $x + y + z = 3, y - z = 0.$

Solution. Let $z = t$ where t is a parameter

$\Rightarrow y = t \qquad (\because y = z)$

also $x + y + z = 3$

$\Rightarrow x = 3 - y - z = 3 - t - t$

$\Rightarrow x = 3 - 2t$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = (3 - 2t)\hat{\imath} + t\hat{\jmath} + t\hat{k}$ **Answer.**

Problem 3. $y^2 + z^2 = 9, x = 9 - y^2,\ y = 3\sin t$.

Solution. Since $y = 3\sin t$

$\Rightarrow x = 9 - (3\sin t)^2$

$\Rightarrow x = 9(1 - sin^2 t) = 9cos^2 t$

also $z^2 = 9 - y^2 \quad = 9 - (3\sin t)^2 \quad = 9(1 - sin^2 t) \quad = 9cos^2 t$

$\Rightarrow z = \pm 3\cos t$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = (9cos^2 t)\hat{\imath} + (3\sin t)\hat{\jmath} \pm (3\cos t)\hat{k}$ **Answer.**

Problem 4. $y^2 = x^2 + z^2, y = 2,\ x = 2\sin t$.

Solution. Since $x = 2\sin t$ and $y = 2$

$z^2 = y^2 - x^2$

$\Rightarrow z^2 = 4 - (2\sin t)^2 \quad = 4(1 - sin^2 t) \quad = 4cos^2 t$

$\Rightarrow z = \pm 2\cos t$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = (2\sin t)\hat{\imath} + 2\hat{\jmath} \pm (2\cos t)\hat{k}$ **Answer.**